A Paley-Wiener theorem for the Bessel-Laplace transform (I): The case $SU(n,n)/GL(n,C)\_+$

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A PALEY-WIENER THEOREM FOR THE BESSEL-LAPLACE TRANSFORM (I): THE CASE $SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$

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Abstract. Let $\mathfrak{q}$ be the tangent space to the noncompact causal symmetric space $SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$ at the origin. In this paper we give an explicit formula for the Bessel functions on $\mathfrak{q}$, and we then use it to prove a Paley-Wiener theorem for the Bessel-Laplace transform on $\mathfrak{q}$. Further, an Abel transform for $\mathfrak{q}$ is defined and inverted.

1. Introduction

In [33] Paley and Wiener proved that the image of the space $\mathcal{L}^2([-R,R])$ by the Euclidean Fourier transform is the space of holomorphic functions F on $\mathbb{C}$ such that $F_{|i\mathbb{R}}\in\mathcal{L}^2(i\mathbb{R})$ and $\sup_{\lambda\in\mathbb{C}} e^{-R|\Re(\lambda)|}|F(\lambda)|<\infty$.

This kind of theorem has been generalized to several different settings. We may cite the following situations: For the spherical Fourier transform on Riemannian symmetric spaces of the noncompact type, a Paley-Wiener type theorem was investigated by Helgason in [18] and Gangolli in [16]. Lately, the case of Riemannian symmetric spaces of the compact type was done by Branson, Ólafsson, and Pasquale in [9]. Helgason-Gangolli's Paley-Wiener theorem was generalized later by Opdam in [30] for the so-called Cherednik transform. Another direction has been attempted to extend the theory of Paley-Wiener type theorems to the setting of noncompact causal symmetric spaces. This was done first by Andersen and Ólafsson in [3] for the rank-one case. The extension to noncompact causal symmetric spaces of Cayley type was given in [4, 5]. Later, Andersen, Ólafsson, and Schlichtkrull in [2] and Ólafsson and Pasquale in [32] established a Paley-Wiener theorem for the spherical Laplace transform on noncompact causal symmetric spaces with even multiplicities. See [2] (or [32]) for the complete list of the so-called causal symmetric spaces with even multiplicities.

Another important setting is that of integral transforms on flat symmetric spaces. In [19] Helgason considered the tangent space to a noncompact Riemannian symmetric space at the origin, and he proved a Paley-Wiener theorem for the Bessel-Fourier transform on the tangent space. This result was generalized later by de Jeu in [26] to the so-called Dunkl transform.

Of course the above list of situations where the Paley-Wiener theorem was generalized is far from being complete. See also [1, 6, 7].

In the present paper we consider the tangent space, say $\mathfrak{q}$, to the causal symmetric space $SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$ at the origin, and we characterize the image of a certain class of smooth functions on $\mathfrak{q}$ by what we shall call the Bessel-Laplace transform (Theorem A). As tools for this study we prove an explicit formula for the Bessel functions on $\mathfrak{q}$, and we investigate some properties of the Bessel-Laplace transform. To establish the first tool, our approach uses the explicit formula of the spherical functions on

$SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$ proved in [4], by taking an appropriate zero-curvature limit. We mention that the contraction process has been carried out by several authors in different settings. See for instance [25, 12, 34, 14, 35, 10, 11].

In the last section we introduce an Abel transform for $\mathfrak{q}$, and we obtain explicitly its inversion formula (Theorem B). The study of the Abel transform and its inversion formula for the tangent spaces to Riemannian symmetric spaces at the origin is the subject of several papers. See for instance [20, 28, 39].

2. Notation and Background

Let $\underline{G}=SU(n,n)$ be the group of complex matrices with determinant 1 which preserve the Hermitian form

$$z_1\overline{w}_1+\cdots+z_n\overline{w}_n-z_{n+1}\overline{w}_{n+1}-\cdots-z_{2n}\overline{w}_{2n},$$

for $z,w\in\mathbb{C}^{2n}$. The group $\underline{G}$ is a connected noncompact semi-simple Lie group with finite center. Its Lie algebra $\underline{\mathfrak{g}}=\mathfrak{su}(n,n)$ is given by

$$\underline{\mathfrak{g}}=\left\{\begin{bmatrix} a & b\\ b^* & c\end{bmatrix}\mid a=-a^*,\ c=-c^*,\ \mathrm{tr}(a+c)=0\right\},$$

where $a,b,c\in M(n,\mathbb{C})$. It is well known that $\underline{\mathfrak{g}}$ is isomorphic to the Lie algebra

$$\mathfrak{g}:=\left\{\begin{bmatrix} \alpha & \beta\\ \gamma & -\alpha^*\end{bmatrix}\mid \beta=\beta^*,\ \gamma=\gamma^*,\ \Im(\mathrm{tr}(\alpha))=0\right\}.$$

Denote by G the analytic subgroup of $GL(2n,\mathbb{C})$ with Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into the (±1)-eigenspaces of the Cartan involution $\theta(X):=-X^*$, with $X\in\mathfrak{g}$. More precisely

$$\mathfrak{k}=\left\{\begin{bmatrix} \alpha & \beta\\ -\beta & \alpha\end{bmatrix}\mid \alpha+\alpha^*=0,\ \beta=\beta^*,\ \Im(\mathrm{tr}(\alpha))=0\right\},$$

and

$$\mathfrak{p}=\left\{\begin{bmatrix} \alpha & \beta\\ \beta & -\alpha\end{bmatrix}\mid \alpha=\alpha^*,\ \beta=\beta^*\right\}.$$

The analytic subgroup K of G with Lie algebra $\mathfrak{k}$ is isomorphic to $S(U(n)\times U(n))$. The quotient $\mathcal{M}^{\mathrm{d}}:=G/K$ is a Riemannian symmetric space of the non-compact type.

Set $\mathfrak{h}:=\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{R}\cong\{\alpha\in\mathfrak{gl}(n,\mathbb{C})\mid\Im(\mathrm{tr}(\alpha))=0\}$. We may embed $\mathfrak{h}$ in $\mathfrak{g}$ as follows

$$\mathfrak{h}\ni\alpha\hookrightarrow\begin{bmatrix} \alpha & 0\\ 0 & -\alpha^*\end{bmatrix}\in\mathfrak{g}.$$

In particular, the subalgebra $\mathfrak{h}$ corresponds to the $(+1)$-eigenspace of the involution $\sigma:\mathfrak{g}\to\mathfrak{g}$ defined by

$$\sigma\Big(\begin{bmatrix} \alpha & \beta\\ \gamma & -\alpha^*\end{bmatrix}\Big):=\begin{bmatrix} \alpha & -\beta\\ -\gamma & -\alpha^*\end{bmatrix}.$$

The (-1)-eigenspace $\mathfrak{q}$ of σ is given by

$$\mathfrak{q}=\left\{\begin{bmatrix} 0 & \beta\\ \gamma & 0\end{bmatrix}\mid \beta=\beta^*,\ \gamma=\gamma^*\right\}.$$

Thus $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{q}$ is the σ-eigenspace decomposition of $\mathfrak{g}$. Denote by H the analytic subgroup of G with Lie algebra $\mathfrak{h}$. The quotient $\mathcal{M}:=G/H\cong SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$ is a noncompact causal symmetric space of Cayley type. We refer the reader to [22, Chap. 3] for more details on the theory of causal symmetric spaces of Cayley type. The symmetric space $\mathcal{M}^{\mathrm{d}}$ is (isomorphic to) the so-called Riemannian dual of $\mathcal{M}$.

Let $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ be the Cartan subspace given by

$$\mathfrak{a} := \left\{ a_{\mathbf{t}} = \begin{bmatrix} 0 & \mathbf{t} \\ \mathbf{t} & 0 \end{bmatrix} \mid \mathbf{t} := \mathrm{diag}(t_1/2, \ldots, t_n/2),\ t_1, \ldots, t_n \in \mathbb{R} \right\}.$$

Note that $\mathfrak{a}$ is also a Cartan subspace of $\mathfrak{p}$. From now on we will identify $\mathfrak{a}$ with $\mathbb{R}^n$ via the map

$$\mathbb{R}^n \ni \mathbf{t} \mapsto a_{\mathbf{t}} \in \mathfrak{a}.$$

For $1 \leq i \leq n$, let $\alpha_i \in \mathfrak{a}^*$ be defined by $\alpha_i(\mathbf{t}) = -t_i$. Thus, the roots of $(\mathfrak{g}, \mathfrak{a})$ are given by the long ones $\pm\alpha_i$ $(1 \leq i \leq n)$ and the short ones $\pm(\alpha_j \pm \alpha_i)/2$ $(1 \leq i < j \leq n)$, with multiplicities $m_{\alpha_i} = 1$ and $m_{(\alpha_j \pm \alpha_i)/2} = 2$. The root system $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a})$ is of type C_n. Choose an ordering on Σ such that the set Σ^+ of positive roots is given by

$$\Sigma^+ = \left\{ \alpha_i\ (1 \leq i \leq n), \frac{(\alpha_j \pm \alpha_i)}{2}\ (1 \leq i < j \leq n) \right\}.$$

Then the negative Weyl chamber is given by

$$\mathfrak{a}_- = \{ \mathbf{t} \in \mathbb{R}^n \mid 0 < t_1 < \cdots < t_n \}.$$

Denote by

$$\Sigma_\circ := \left\{ \frac{\pm(\alpha_j - \alpha_i)}{2}\ (1 \leq i < j \leq n) \right\},$$

and let

$$\Sigma_\circ^+ := \Sigma^+ \cap \Sigma_\circ = \left\{ \frac{(\alpha_j - \alpha_i)}{2}\ (1 \leq i < j \leq n) \right\}.$$

The Weyl groups of Σ and $\Sigma_\circ$ are respectively $\mathcal{W} \cong \mathbb{S}_n \times \{\pm 1\}^n$ and $\mathcal{W}_\circ \cong \mathbb{S}_n$, where $\mathbb{S}_n$ is the permutation group of n elements. More precisely, $\mathcal{W}$ acts on $\mathfrak{a}$ by $\mathbf{t} \to (\tau_1 t_{\sigma(1)}, \ldots, \tau_n t_{\sigma(n)})$ with $\tau_i = \pm 1$ and $\sigma \in \mathbb{S}_n$, while $\mathcal{W}_\circ$ acts on $\mathfrak{a}$ by $\mathbf{t} \to (t_{\sigma(1)}, \ldots, t_{\sigma(n)})$.

For all $\boldsymbol{\lambda} \in \mathbb{C}^n$, denote by $\varphi_{\boldsymbol{\lambda}}$ the Harish-Chandra spherical functions on $\mathcal{M}^{\mathrm{d}}$ with spectral $\boldsymbol{\lambda}$ (cf. [17], [21, Chap. IV]). In particular, if we use the identification of functions on $\mathcal{M}^{\mathrm{d}}$ with right K-invariant functions on G, then $\varphi_{\boldsymbol{\lambda}}(kgk') = \varphi_{\boldsymbol{\lambda}}(g)$ for all $k, k' \in K$ and $g \in G$. Thus, the spherical functions are completely determined on the radial part $\exp(\mathfrak{a}_-)$. Furthermore, they are $\mathcal{W}$-invariant on the spectral parameter $\boldsymbol{\lambda}$. In [8] Berezin and Karpelevič gave (without proof) an explicit formula for the Harish-Chandra spherical functions on $SU(n,n)/S(U(n) \times U(n))$. A complete proof can be found in [23].

Theorem 2.1. (cf. [8, 23]) *There exists a constant that depends only on n such that the spherical functions $\varphi_{\boldsymbol{\lambda}}$ on $SU(n,n)/S(U(n) \times U(n))$ are given by*

$$\varphi_{\boldsymbol{\lambda}}(\exp(\mathbf{t})) = \text{const.} \frac{\det\limits_{1 \leq i,j \leq n} \left({}_2F_1\big(\frac{1}{2}(-\lambda_i + \frac{1}{2}), \frac{1}{2}(\lambda_i + \frac{1}{2}); 1; -\mathrm{sh}^2\, t_j\big) \right)}{\prod\limits_{1 \leq i < j \leq n} (\lambda_j^2 - \lambda_i^2) \prod\limits_{1 \leq i < j \leq n} (\mathrm{ch}\, t_j - \mathrm{ch}\, t_i)},$$

for all $\boldsymbol{\lambda} \in \mathbb{C}^n$ such that $\prod_{\alpha \in \Sigma^+} \langle \alpha, \boldsymbol{\lambda} \rangle \neq 0$, and for all $\mathbf{t} \in \mathfrak{a}_-$.

***Remark* 2.2.** For fixed $\mathbf{t}$, the function $\boldsymbol{\lambda} \mapsto \varphi_{\boldsymbol{\lambda}}(\exp(\mathbf{t}))$ has a holomorphic extension to $\mathbb{C}^n$.

For $\boldsymbol{\lambda} \in \mathbb{C}^n$, the spherical Fourier transform $\mathscr{F}^{\mathrm{d}}$ on $\mathcal{M}^{\mathrm{d}}$ can be written for every $f \in \mathcal{C}_c^\infty(\mathfrak{a})^{\mathcal{W}}$ as

$$\mathscr{F}^{\mathrm{d}}(f)(\boldsymbol{\lambda}) = \int_{\mathfrak{a}_-} f(\mathbf{t})\varphi_{-\boldsymbol{\lambda}}(\exp(\mathbf{t}))\Delta(\mathbf{t})d\mathbf{t},$$

where

$$\Delta(\mathbf{t}) = 2^{n(n-1)} \prod_{j=1}^{n} \operatorname{sh} t_j \prod_{1\le i<j\le n} (\operatorname{ch} t_j - \operatorname{ch} t_i)^2. \tag{2.1}$$

The inversion formula for $\mathscr{F}^{\mathrm{d}}$ is given by

$$f(\mathbf{t}) = \text{const.} \int_{i\mathbb{R}^n} \mathscr{F}^{\mathrm{d}}(f)(\boldsymbol{\lambda})\varphi_{\boldsymbol{\lambda}}(\exp(\mathbf{t})) \frac{d\boldsymbol{\lambda}}{|c^{\mathrm{d}}(\boldsymbol{\lambda})|^2}, \qquad \mathbf{t} \in \mathbb{R}^n, \tag{2.2}$$

where

$$c^{\mathrm{d}}(\boldsymbol{\lambda}) = c(\mathrm{d}) \prod_{i=1}^{n} \frac{\Gamma(-\lambda_i)}{\Gamma(-\lambda_i + 1/2)} \prod_{1\le i<j\le n} (\lambda_j^2 - \lambda_i^2)^{-1}. \tag{2.3}$$

The constant "const." is positive and depends only on the normalization of the measures, and $c(\mathrm{d})$ is a positive constant which can be determined from $c^{\mathrm{d}}(\rho) = 1$, where $\rho = (1/2, 3/2, 1/2 + n - 1)$. For more details on the theory of spherical Fourier transforms, we refer to [21, Chap. IV].

Let $c_{\max}$ be the maximal closed cone in $\mathfrak{a}$ ($\cong \mathbb{R}^n$) defined by

$$c_{\max} := \{\mathbf{t} \in \mathbb{R}^n \mid t_i \ge 0 \ (1 \le i \le n)\}.$$

The subset $C_{\max} := \mathrm{Ad}(H)c_{\max} \subset \mathfrak{q}$ is a maximal H-invariant regular cone in $\mathfrak{q}$. Denote by $\Gamma(C_{\max}) := \exp(C_{\max})H$ the semi-group in $SU(n,n)$ with interior $\Gamma(C^\circ_{\max}) = \exp(C^\circ_{\max})H = H\exp(c^\circ_{\max})H$.

For $\boldsymbol{\lambda} \in \mathbb{C}^n$, set $\psi_{\boldsymbol{\lambda}}$ to be the spherical function on $\mathcal{M}$ with spectral $\boldsymbol{\lambda}$ (cf. [15]). Note that $\psi_{\boldsymbol{\lambda}}$ are only defined on $\Gamma(C^\circ_{\max})$, and H-bi-invariant functions. We mention that for an arbitrary noncompact causal symmetric space, the spherical functions are defined in [15] by an integral formula over H. In [27], the authors determine the exact set $\mathcal{E}$ of $\boldsymbol{\lambda}$ for which the integral is finite. Further, a Harish-Chandra expansion type formula for $\psi_{\boldsymbol{\lambda}}$ can be found in [31]. We also note that $\psi_{w\boldsymbol{\lambda}} = \psi_{\boldsymbol{\lambda}}$ for all $w \in \mathcal{W}_\circ$.

In view of the Berezin-Karpelevič formula for $\varphi_{\boldsymbol{\lambda}}$, and Ólafsson's expansion for $\psi_{\boldsymbol{\lambda}}$, in [4] the authors proved the following statement.

Theorem 2.3. (cf. [4]) *There exists a constant that depends only on n such that the spherical functions $\psi_{\boldsymbol{\lambda}}$ on $SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}^*_+$ are given by*

$$\psi_{\boldsymbol{\lambda}}(\exp(\mathbf{t})) = \text{const.} \frac{\det\limits_{1\le i,j\le n} \Big(Q_{\lambda_i - 1/2}(\operatorname{ch} t_j)\Big)}{\prod\limits_{1\le i<j\le n} (\lambda_j^2 - \lambda_i^2) \prod\limits_{1\le i<j\le n} (\operatorname{ch} t_j - \operatorname{ch} t_i)},$$

for all $\boldsymbol{\lambda} \in \mathbb{C}^n$ such that $\Re(\lambda_i) > 0$ $(1 \le i \le n)$ and for all $\mathbf{t} \in \mathfrak{a}_-$. Here Q_μ denotes the Legendre function of the second kind.

***Remark* 2.4.** Recall the set $\mathcal{E}$ from [27]. In the $SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}^*_+$-case, we have

$$\mathcal{E} = \{\ \boldsymbol{\lambda} \in \mathbb{C}^n \mid \Re(\lambda_i) > -1/2 \ (1 \le i \le n),\ \Re(\lambda_i + \lambda_j) > 0 \ (1 \le i \ne j \le n)\ \}.$$

Thus, the statement of Theorem 2.3 remains valid for every $\boldsymbol{\lambda}$ in $\mathcal{E}$. On the other hand, using [24, Theorem 1.2.4] and the fact that $\nu \mapsto Q_\nu(z)$ is a meromorphic function on $\mathbb{C}$ with poles at the points $\nu \in -\mathbb{N}^*$, one can see that for fixed $\mathbf{t}$, the function $\boldsymbol{\lambda} \mapsto \psi_{\boldsymbol{\lambda}}(\exp(\mathbf{t}))$ has a meromorphic extension to $\mathbb{C}^n$ with simple poles at $\boldsymbol{\lambda} \in \mathbb{C}^n$ such that $\lambda_i \in -\mathbb{N}^* + 1/2$ $(1 \leq i \leq n)$ and $\lambda_i + \lambda_j = 0$ $(1 \leq i \neq j \leq n)$.

We may identify the space $\mathcal{C}_c^\infty(H \backslash \Gamma(C_{\max}^\circ)/H)$ with $\mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$. Thus, the spherical Laplace transform of all $f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$ can be written as

$$\mathscr{L}(f)(\boldsymbol{\lambda}) = \int_{\mathfrak{a}_-} f(\mathbf{t})\psi_{\boldsymbol{\lambda}}(\exp \mathbf{t})\Delta(\mathbf{t})d\mathbf{t},$$

where $\Delta(\mathbf{t})$ is given by (2.1). The inverse spherical Laplace transform is given by

$$f(\mathbf{t}) = \text{const.} \int_{i\mathbb{R}^n} \mathscr{L}(f)(\boldsymbol{\lambda})\varphi_{\boldsymbol{\lambda}}(\exp(\mathbf{t}))\frac{d\boldsymbol{\lambda}}{c(\boldsymbol{\lambda})c^{\mathrm{d}}(-\boldsymbol{\lambda})}, \qquad \mathbf{t} \in c_{\max}^\circ \tag{2.4}$$

where c^{d} is given by (2.3), and

$$c(\boldsymbol{\lambda}) = c(\Omega)\prod_{i=1}^n \frac{\Gamma(\lambda_i + 1/2)}{\Gamma(\lambda_i + 1)} \prod_{1 \leq i < j \leq n} (\lambda_j^2 - \lambda_i^2)^{-1}. \tag{2.5}$$

Here $c(\Omega)$ is a positive constant, see [27, Theorem III.5]. We refer to [15] and [22, Chap. 8] for more details on the theory of spherical Laplace transforms.

3. The Bessel-Laplace transform

Recall that $\mathcal{M}^{\mathrm{d}} = SU(n,n)/S(U(n) \times U(n))$ and $\mathcal{M} = SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$. For $\epsilon > 0$, write $g_\epsilon = k\exp(\epsilon X)$ with $k \in K$ and $X \in \mathfrak{p}$. Denote by $\Phi(\boldsymbol{\lambda}, X) := \lim_{\epsilon \to 0} \varphi_{\boldsymbol{\lambda}/\epsilon}(g_\epsilon)$. In [28] the author proved that the limit $\Phi(\boldsymbol{\lambda}, X)$ exists and is a smooth function. The limiting functions are the so-called Bessel functions on the flat symmetric space $\mathfrak{p}$. In [14, 10] this result was generalized to arbitrary noncompact Riemannian symmetric space. Even more, in [11] a similar result (for arbitrary noncompact causal symmetric space) was proved when $\varphi_{\boldsymbol{\lambda}}$ is replaced by the spherical function $\psi_{\boldsymbol{\lambda}}$. More precisely, if $\gamma_\epsilon = \exp(\epsilon X)h$ with $X \in C_{\max}^\circ$ and $h \in H$, then, for certain $\boldsymbol{\lambda} \in \mathfrak{a}_{\mathbb{C}}^*$, the limit $\Psi(\boldsymbol{\lambda}, X) := \lim_{\epsilon \to 0} \psi_{\boldsymbol{\lambda}/\epsilon}(\gamma_\epsilon)$ and its derivatives exist. We refer to [14, 10, 11] for more details.

Theorem 3.1. (i) (cf. [28]) *For all $\boldsymbol{\lambda} \in \mathbb{C}^n$ such that $\prod_{\alpha \in \Sigma^+}\langle \boldsymbol{\lambda}, \alpha\rangle \neq 0$, and for all $\mathbf{t} \in \mathfrak{a}_-$, there exists a constant which depends only on n such that*

$$\Phi(\boldsymbol{\lambda}, \mathbf{t}) = \text{const.} \frac{\det\limits_{1 \leq i,j \leq n}\Big(I_0(\lambda_i t_j)\Big)}{\prod\limits_{1 \leq i < j \leq n}(\lambda_j^2 - \lambda_i^2)\prod\limits_{1 \leq i < j \leq n}(t_j^2 - t_i^2)},$$

where $I_\nu(z) := e^{-i\nu\pi/2}J_\nu(iz)$ and J_ν the Bessel function of the first kind

$$J_\nu(z) := \sum_{\ell=0}^\infty (-1)^\ell \frac{(z/2)^{2\ell+\nu}}{\Gamma(\ell + \nu + 1)\ell!}.$$

The Bessel function Φ extends to a holomorphic function on $\mathbb{C}^n \times \mathbb{C}^n$.

(ii) *For all* $\boldsymbol{\lambda} \in \mathbb{C}^n$ *such that* $\Re(\lambda_i) > 0$ *(*$1 \le i \le n$*), and for all* $\mathbf{t} \in \mathfrak{a}_-$*, there exists a constant which depends only on* n *such that*

$$\Psi(\boldsymbol{\lambda}, \mathbf{t}) = \text{const.} \frac{\det\limits_{1\le i,j\le n} \Big(K_0(\lambda_i t_j)\Big)}{\prod\limits_{1\le i<j\le n} (\lambda_j^2 - \lambda_i^2) \prod\limits_{1\le i<j\le n} (t_j^2 - t_i^2)},$$

where

$$K_0(z) := \lim_{\nu\to 0} \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}$$

denotes the Bessel function of the third kind. For fixed $\mathbf{t}$*, the function* $\boldsymbol{\lambda} \mapsto \Psi(\boldsymbol{\lambda}, \mathbf{t})$ *has a meromorphic extension to*

$$\mathfrak{D} = \{\, \boldsymbol{\lambda} \in \mathbb{C}^n \mid \lambda_i \in \mathbb{C}\backslash] - \infty, 0] \,\},$$

with simple poles at $\boldsymbol{\lambda} \in \mathfrak{D}$ *such that* $\lambda_i + \lambda_j = 0$ *for some* $1 \le i \ne j \le n$*.*

Proof. (ii) For $\epsilon > 0$, write $\psi_{\boldsymbol{\lambda}/\epsilon}(\exp(\epsilon\mathbf{t}))$ as

$$\begin{aligned}\psi_{\boldsymbol{\lambda}/\epsilon}(\exp(\epsilon\mathbf{t})) \quad &= \quad \text{const.} \frac{\epsilon^{n(n-1)}}{\prod\limits_{1\le i<j\le n} (\lambda_j^2 - \lambda_i^2) \prod\limits_{1\le i<j\le n} (\operatorname{sh}^2(\epsilon t_j/2) - \operatorname{sh}^2(\epsilon t_i/2))} \times \\ &\qquad \sum_{\sigma\in\mathbb{S}_n} (-1)^\sigma \prod_{i=1}^n Q_{\lambda_{\sigma(i)}/\epsilon - 1/2}(\operatorname{ch}(\epsilon t_i)).\end{aligned}$$

By [36, p.259], we have

$$\begin{aligned}&(\operatorname{sh} t)^{-\mu} \frac{\Gamma(\lambda - \mu + 1/2)}{\Gamma(\lambda + \mu + 1/2)} Q^\mu_{\lambda - 1/2}(\operatorname{ch} t) \\ &= \frac{e^{i\pi\mu}}{2} \Bigg\{ \frac{\Gamma(-\mu)}{2^\mu} {}_2F_1\Big(\frac{1}{2}(\lambda + \mu + \frac{1}{2}), \frac{1}{2}(- \lambda + \mu + \frac{1}{2}); 1 + \mu; - \operatorname{sh}^2 t\Big) + \\ &\quad \frac{\Gamma(\mu)}{2^{-\mu}} (\operatorname{sh} t)^{-2\mu} \frac{\Gamma(\lambda - \mu + 1/2)}{\Gamma(\lambda + \mu + 1/2)} {}_2F_1\Big(\frac{1}{2}(\lambda - \mu + \frac{1}{2}), \frac{1}{2}(- \lambda - \mu + \frac{1}{2}); 1 - \mu; - \operatorname{sh}^2 t\Big) \Bigg\}.\end{aligned}$$

Using the well known formula

$$\frac{\Gamma(z + a)}{\Gamma(z + b)} = z^{a-b}(1 + O(z^{-1})) \qquad \text{as} \quad z \to \infty, \tag{3.1}$$

together with the hypergeometric series of ${}_2F_1$, we obtain:

$$\lim_{\epsilon\to 0} {}_2F_1\Big(\frac{1}{2}(\frac{\lambda}{\epsilon} \pm \mu + \frac{1}{2}), \frac{1}{2}(- \frac{\lambda}{\epsilon} \pm \mu + \frac{1}{2}); 1 \pm \mu; - \operatorname{sh}^2(\epsilon t)\Big) = \Gamma(\pm\mu + 1)\Big(\frac{\lambda t}{2}\Big)^{\mp\mu} I_{\pm\mu}(\lambda t),$$

and

$$\lim_{\epsilon\to 0} \frac{\Gamma(\frac{\lambda}{\epsilon} - \mu + \frac{1}{2})}{\Gamma(\frac{\lambda}{\epsilon} + \mu + \frac{1}{2})} (\operatorname{sh}(\epsilon t))^{-2\mu} = (\lambda t)^{-2\mu}.$$

Here I_μ denotes the modified Bessel function given in the statement (i) above. Thus

$$\begin{aligned}
&\lim_{\epsilon\to 0}(\operatorname{sh}\epsilon t)^{-\mu}\frac{\Gamma(\lambda/\epsilon-\mu+1/2)}{\Gamma(\lambda/\epsilon+\mu+1/2)}Q^{\mu}_{\lambda/\epsilon-1/2}(\operatorname{ch}(\epsilon t))\\
&=\frac{e^{i\pi\mu}}{2}\Big\{\frac{\Gamma(-\mu)\Gamma(1+\mu)}{2^{\mu}}\Big(\frac{\lambda t}{2}\Big)^{-\mu}I_\mu(\lambda t)+\frac{\Gamma(\mu)\Gamma(1-\mu)}{2^{-\mu}}\frac{(\lambda t)^{-\mu}}{2^{\mu}}I_{-\mu}(\lambda t)\Big\}\\
&=e^{i\pi\mu}(\lambda t)^{-\mu}\Big\{\frac{\pi}{2}\frac{I_{-\mu}(\lambda t)-I_\mu(\lambda t)}{\sin(\pi\mu)}\Big\},
\end{aligned}$$

and therefore

$$\lim_{\epsilon\to 0}Q_{\lambda/\epsilon-1/2}(\operatorname{ch}(\epsilon t))=\lim_{\mu\to 0}\frac{\pi}{2}\frac{I_{-\mu}(\lambda t)-I_\mu(\lambda t)}{\sin(\pi\mu)}=K_0(\lambda t).$$

To prove that, for fixed $\mathbf{t}$, the function $\boldsymbol{\lambda}\mapsto\Psi(\boldsymbol{\lambda},\mathbf{t})$ is meromorphic on $\mathfrak{D}$ with simple poles at $\lambda_i+\lambda_j=0$ for $i\neq j$, one can proceed as follows: It is well known that $K_\nu(z)$ is an analytic function of z for all $z\in\mathbb{C}\backslash]\infty,0]$. Further, it is clear that $\lambda_{i_0}+\lambda_{j_0}=0$, with $i_0\neq j_0$, is a simple pole for $\Psi(\boldsymbol{\lambda},\cdot)$. Further, write

$$\det_{1\le i,j\le n}\Big(K_0(\lambda_i t_j)\Big)=(\lambda_{j_0}-\lambda_{i_0})\begin{vmatrix}
K_0(\lambda_1t_1) & \cdots & K_0(\lambda_1t_n)\\
\vdots & & \vdots\\
K_0(\lambda_{i_0}t_1) & \cdots & K_0(\lambda_{i_0}t_n)\\
\vdots & & \vdots\\
K_0(\lambda_{j_0-1}t_1) & \cdots & K_0(\lambda_{j_0-1}t_n)\\
\frac{K_0(\lambda_{j_0}t_1)-K_0(\lambda_{i_0}t_1)}{\lambda_{j_0}-\lambda_{i_0}} & \cdots & \frac{K_0(\lambda_{j_0}t_n)-K_0(\lambda_{i_0}t_n)}{\lambda_{j_0}-\lambda_{i_0}}\\
K_0(\lambda_{j_0+1}t_1) & \cdots & K_0(\lambda_{j_0+1}t_n)\\
\vdots & & \vdots\\
K_0(\lambda_nt_1) & \cdots & K_0(\lambda_nt_n)
\end{vmatrix}.$$

Thus,

$$\lim_{\lambda_{j_0}\to\lambda_{i_0}}\frac{\det_{1\le i,j\le n}\Big(K_0(\lambda_it_j)\Big)}{\prod_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2)}=\frac{1}{(2\lambda_{i_0})\prod_{\substack{1\le i<j\le n\\ i\neq i_0}}(\lambda_j^2-\lambda_i^2)\prod_{\substack{j=i_0+1\\ j\neq j_0}}^{n}(\lambda_j^2-\lambda_{i_0}^2)}\times
\begin{vmatrix}
K_0(\lambda_1t_1) & \cdots & K_0(\lambda_1t_n)\\
\vdots & & \vdots\\
K_0(\lambda_{i_0}t_1) & \cdots & K_0(\lambda_{i_0}t_n)\\
\vdots & & \vdots\\
K_0(\lambda_{j_0-1}t_1) & \cdots & K_0(\lambda_{j_0-1}t_n)\\
-t_1K_1(\lambda_{i_0}t_1) & \cdots & -t_nK_1(\lambda_{i_0}t_n)\\
K_0(\lambda_{j_0+1}t_1) & \cdots & K_0(\lambda_{j_0+1}t_n)\\
\vdots & & \vdots\\
K_0(\lambda_nt_1) & \cdots & K_0(\lambda_nt_n)
\end{vmatrix}.$$

Moreover, if $\mathbf{a} = (a, a, \dots, a) \in \mathfrak{D}$, then

$$\lim_{\boldsymbol{\lambda}\to\mathbf{a}} \frac{\det\limits_{1\le i,j\le n}\Big(K_0(\lambda_i t_j)\Big)}{\prod\limits_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2)} = (2a)^{\frac{-n(n-1)}{2}} \det_{1\le i,j\le n}\Big(t_i^{j-1}K_0^{(j-1)}(at_i)\Big).$$

□

***Remark* 3.2.** (i) The Bessel function Φ is symmetric in its arguments. Further, since $I_0(z)$ is an even function, clearly we have $\Phi(w\boldsymbol{\lambda}, \mathbf{t}) = \Phi(\boldsymbol{\lambda}, w\mathbf{t}) = \Phi(\boldsymbol{\lambda}, \mathbf{t})$ for all $w \in \mathcal{W} = \mathbb{S}_n \times \{\pm 1\}^n$. For general results in the theory of Bessel functions on Cartan motion groups, we refer to [21, 29, 38, 10, 11].

(ii) The Bessel function Ψ is symmetric in $\boldsymbol{\lambda}$ and $\mathbf{t}$, with $\Psi(w_0\boldsymbol{\lambda}, \mathbf{t}) = \Psi(\boldsymbol{\lambda}, w_0\mathbf{t}) = \Psi(\boldsymbol{\lambda}, \mathbf{t})$ for all $w_0 \in \mathcal{W}_\circ = \mathbb{S}_n$.

Following [20], the Bessel-Fourier transform $\widetilde{\mathscr{F}}^{\mathrm{d}}$ on the flat symmetric space $\mathfrak{p}$ is given for any function $f \in \mathcal{C}_c^\infty(\mathfrak{a})^{\mathcal{W}}$ by

$$\widetilde{\mathscr{F}}^{\mathrm{d}}(f)(\boldsymbol{\lambda}) = \int_{\mathfrak{a}_-} f(\mathbf{t})\Phi(\boldsymbol{\lambda}, \mathbf{t})\omega(\mathbf{t})d\mathbf{t},$$

where

$$\omega(\mathbf{t}) := \prod_{i=1}^n t_i \prod_{1\le i<j\le n}(t_j^2 - t_i^2)^2, \qquad \mathbf{t} \in \mathfrak{a}_-. \tag{3.2}$$

Further, there exists a positive constant depending only on the normalization of the measures such that

$$f(\mathbf{t}) = \text{const.} \int_{i\mathbb{R}^n} \widetilde{\mathscr{F}}^{\mathrm{d}}(f)(\boldsymbol{\lambda})\Phi(\boldsymbol{\lambda}, \mathbf{t})\omega(\boldsymbol{\lambda})d\boldsymbol{\lambda}, \tag{3.3}$$

where

$$\omega(\boldsymbol{\lambda}) := \prod_{i=1}^n |\lambda_i| \prod_{1\le i<j\le n} |\lambda_j^2 - \lambda_i^2|^2. \tag{3.4}$$

Observe that one may recover the definition of $\widetilde{\mathscr{F}}^{\mathrm{d}}$ and its inversion formula via $\mathscr{F}^{\mathrm{d}}$, by applying the limit transition approach. Indeed, for $\epsilon > 0$, set $f_\epsilon(\mathbf{t}) := f(\epsilon^{-1}\mathbf{t})$. Thus

$$\begin{aligned}
&\mathscr{F}^{\mathrm{d}}(f_\epsilon)(\boldsymbol{\lambda}/\epsilon)\\
&= \int_{\mathfrak{a}_-} f_\epsilon(\mathbf{t})\varphi_{-\boldsymbol{\lambda}/\epsilon}(\exp\mathbf{t})\prod_{i=1}^n \mathrm{sh}(t_i) \prod_{1\le i<j\le n}(2\,\mathrm{ch}(t_j) - 2\,\mathrm{ch}(t_i))^2 d\mathbf{t}\\
&\sim \epsilon^{n(2n-1)} \int_{\mathfrak{a}_-} f(\mathbf{t})\varphi_{-\boldsymbol{\lambda}/\epsilon}(\exp\epsilon\mathbf{t})\prod_{i=1}^n t_i \prod_{1\le i<j\le n}(t_j^2 - t_i^2)^2 d\mathbf{t} \qquad \text{as } \epsilon \longrightarrow 0.
\end{aligned}$$

Hence

$$\lim_{\epsilon\to 0} \epsilon^{-n(2n-1)}\mathscr{F}^{\mathrm{d}}(f_\epsilon)(\boldsymbol{\lambda}/\epsilon) = \widetilde{\mathscr{F}}^{\mathrm{d}}(f)(\boldsymbol{\lambda}). \tag{3.5}$$

By virtue of (3.1), one can also use the inversion formula (2.2) for $\mathscr{F}^{\mathrm{d}}$ to obtain (3.3). We should mention that the Bessel-Fourier transform has been carried out by several authors in different settings (see for instance [19, 20, 28, 39]).

***Remark* 3.3.** In [13] Dunkl introduced an integral transformation on the space $\mathcal{L}^2(\mathfrak{a}, d\mu)$ (where μ is some suitable measure) in terms of the eigenfunctions of the so-called Dunkl operators. This class of Dunkl transforms encloses the Bessel-Fourier transforms on flat symmetric spaces.

Define the Bessel-Laplace transform on the flat symmetric space $\mathfrak{q}$ of an element $f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$ by

$$\widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda}) = \int_{\mathfrak{a}_-} f(\mathbf{t})\Psi(\boldsymbol{\lambda}, \mathbf{t})\omega(\mathbf{t})d\mathbf{t}, \qquad \forall f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ},$$

whenever this integral converges. Again, one may obtain the above natural definition of $\widetilde{\mathscr{L}}$ via the spherical Laplace transform $\mathscr{L}$.

By [31, Lemma 4.16], we know that if $f \in \mathcal{C}_c(c_{\max}^\circ)^{\mathcal{W}_\circ}$, then there exists a unique function $f^{\mathrm{d}} \in \mathcal{C}_c(\mathfrak{a})^{\mathcal{W}}$ such that $f^{\mathrm{d}}_{|\mathfrak{a}_-} \equiv f_{|\mathfrak{a}_-}$. Thus, we may obtain a relation between $\widetilde{\mathscr{F}}^{\mathrm{d}}$ and $\widetilde{\mathscr{L}}$ as follows: For $\mathbf{t} \in \mathfrak{a}_-$, we know that

$$\varphi_{\boldsymbol{\lambda}}(\exp(-\mathbf{t})) = \sum_{\tau \in \{\pm 1\}^n} \frac{c^{\mathrm{d}}(\tau\boldsymbol{\lambda})}{c(\tau\boldsymbol{\lambda})} \psi_{\tau\boldsymbol{\lambda}}(\exp(\mathbf{t})), \tag{3.6}$$

for almost every $\boldsymbol{\lambda} \in \mathbb{C}^n$ (cf. [22, Theorem 8.4.4]). Further, in the light of (3.1), we have

$$c^{\mathrm{d}}(\boldsymbol{\lambda}/\epsilon) \sim \epsilon^{n(n-1/2)} c(\mathrm{d}) \prod_{i=1}^n (-\lambda_i)^{-1/2} \prod_{1 \le i < j \le n} (\lambda_j^2 - \lambda_i^2)^{-1} \qquad \text{as } \epsilon \to 0,$$

and

$$c(\boldsymbol{\lambda}/\epsilon) \sim \epsilon^{n(n-1/2)} c(\Omega) \prod_{i=1}^n \lambda_i^{-1/2} \prod_{1 \le i < j \le n} (\lambda_j^2 - \lambda_i^2)^{-1} \qquad \text{as } \epsilon \to 0.$$

Thus

$$\Phi(\boldsymbol{\lambda}, \mathbf{t}) = \frac{c(\mathrm{d})}{c(\Omega)} \sum_{\tau = (\tau_i)_i \in \{\pm 1\}^n} \prod_{i=1}^n \{(-\tau_i\lambda_i)^{-1/2} (\tau_i\lambda_i)^{1/2}\} \Psi(\tau\boldsymbol{\lambda}, \mathbf{t}) \qquad \forall\, \mathbf{t} \in \mathfrak{a}_-, \tag{3.7}$$

for almost every $\boldsymbol{\lambda} \in \mathbb{C}^n$. When $n = 1$, we have $c(\mathrm{d})c(\Omega)^{-1} = \pi^{-1}$, and the equality (3.7) coincides with the well known formula $K_0(z) - K_0(-z) = i\pi I_0(z)$ (cf. [36, p. 428]). Now the following is clear.

Corollary 3.4. *For almost every $\boldsymbol{\lambda} \in \mathbb{C}^n$, we have*

$$\widetilde{\mathscr{F}}^{\mathrm{d}}(f^{\mathrm{d}})(\boldsymbol{\lambda}) = \frac{c(\mathrm{d})}{c(\Omega)} \sum_{\tau = (\tau_i)_i \in \{\pm 1\}^n} \prod_{i=1}^n \{(-\tau_i\lambda_i)^{-1/2} (\tau_i\lambda_i)^{1/2}\} \widetilde{\mathscr{L}}(f)(\tau\boldsymbol{\lambda}), \quad \forall f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}.$$

In particular, the right hand side extends to an analytic function on $\mathbb{C}^n$.

The inversion formula for the transform $\widetilde{\mathscr{L}}$ is now immediate.

Theorem 3.5. *If $f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$, then there exists a positive constant such that*

$$f(\mathbf{t}) = \text{const.} \int_{i\mathbb{R}^n} \widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda}) \Phi(\boldsymbol{\lambda}, \mathbf{t}) \omega(\boldsymbol{\lambda}) \prod_{i=1}^n \frac{\lambda_i}{|\lambda_i|} d\boldsymbol{\lambda}$$

for all $\mathbf{t} \in \mathfrak{a}_-$. Here $\omega(\boldsymbol{\lambda})$ is as in (3.4).

Proof. For $\mathbf{t} \in \mathfrak{a}_-$ we have

$$\begin{aligned} f(\mathbf{t}) &= \text{const.} \int_{i\mathbb{R}^n} \widetilde{\mathscr{F}}^{\mathrm{d}}(f^{\mathrm{d}})(\boldsymbol{\lambda})\Phi(\boldsymbol{\lambda}, \mathbf{t})\omega(\boldsymbol{\lambda})d\boldsymbol{\lambda} \\ &= \text{const.} \int_{i\mathbb{R}^n} \Big\{ \sum_{\tau\in\{\pm1\}^n} (\prod_{i=1}^n \tau_i)\widetilde{\mathscr{L}} f(\tau\boldsymbol{\lambda}) \Big\} \Phi(\boldsymbol{\lambda}, \mathbf{t})\omega(\boldsymbol{\lambda}) \prod_{i=1}^n \frac{\lambda_i}{|\lambda_i|} d\boldsymbol{\lambda}. \end{aligned}$$

The statement is now due to the $\mathcal{W}$-invariance of Φ and $\omega(\boldsymbol{\lambda})d\boldsymbol{\lambda}$. □

4. The Paley-Wiener theorem

For $R > 0$, let $\mathfrak{B}_R := \{\mathbf{t} \in \mathbb{R}^n \mid \|\mathbf{t}\| \leq R\}$. Denote by $\mathcal{C}_R^\infty(\mathfrak{a})$ the space of smooth functions on $\mathfrak{a}$ with support contained in the closed ball $\mathfrak{B}_R$. Define the Paley-Wiener space $\mathcal{H}_{\mathcal{W}}^R(\mathbb{C}^n)$ as the space of $\mathcal{W}$-invariant holomorphic functions on $\mathbb{C}^n$ with the property that for each $M \in \mathbb{N}$ there exists a constant $c_M > 0$ such that

$$|g(\boldsymbol{\lambda})| \leq c_M(1 + \|\boldsymbol{\lambda}\|)^{-M} e^{R\|\Re(\boldsymbol{\lambda})\|},$$

for all $\boldsymbol{\lambda} \in \mathbb{C}^n$. By $\mathcal{H}_{\mathcal{W}}(\mathbb{C}^n)$ we will denote the union of the spaces $\mathcal{H}_{\mathcal{W}}^R(\mathbb{C}^n)$ over all $R > 0$.

Theorem 4.1. (cf. [19]) *The Bessel-Fourier transform $f \mapsto \widetilde{\mathscr{F}}^{\mathrm{d}}(f)$ is a bijection of $\mathcal{C}_c^\infty(\mathfrak{a})^{\mathcal{W}}$ onto $\mathcal{H}_{\mathcal{W}}(\mathbb{C}^n)$. The function f has support in the ball $\mathfrak{B}_R$ if and only if $\widetilde{\mathscr{F}}^{\mathrm{d}}(f) \in \mathcal{H}_{\mathcal{W}}^R(\mathbb{C}^n)$, for all $R > 0$.*

Next we will discuss a Paley-Wiener theorem for $\widetilde{\mathscr{L}}$. For $0 < r < R < \infty$, denote by $\mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$ the space of $\mathcal{W}_\circ$-invariant meromorphic functions g on $\mathfrak{D}$ with at most simple poles at $\lambda_i + \lambda_j = 0$ for some $1 \leq i \neq j \leq n$, such that:

($\mathbb{P}_1$) The map

$$\boldsymbol{\lambda} \mapsto \mathtt{av}(g)(\boldsymbol{\lambda}) := \sum_{\tau\in\{\pm1\}^n} \prod_{i=1}^n \{(-\tau_i\lambda_i)^{-1/2}(\tau_i\lambda_i)^{1/2}\} g(\tau\boldsymbol{\lambda})$$

extends to a function in $\mathcal{H}_{\mathcal{W}}^R(\mathbb{C}^n)$.

($\mathbb{P}_2$) For all $M \in \mathbb{N}$, there exists a constant c_M such that for $\boldsymbol{\lambda} \in \mathfrak{D}$ with $\Re(\lambda_i) \geq 0$ $(1 \leq i \leq n)$ we have

$$\prod_{i=1}^n |\lambda_i|^{1/2} \prod_{1\leq i<j\leq n} |\lambda_i^2 - \lambda_j^2| \, |g(\boldsymbol{\lambda})| \leq c_M(1 + \|\boldsymbol{\lambda}\|)^{-M} e^{-r\langle\Re(\boldsymbol{\lambda}),\mathbf{t}_0\rangle},$$

where $\mathbf{t}_0 := (1, \ldots, 1)$.

Denote by $\mathcal{PW}_\circ(\mathbb{C}^n)$ the union of the spaces $\mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$ over all $0 < r < R < \infty$.

Claim 1. For all $\boldsymbol{\lambda} \in \mathfrak{D}$ such that $\Re(\lambda_i) \geq 0$ $(1 \leq i \leq n)$, and for all $\mathbf{t} \in \mathbb{R}^n$ such that $t_i \geq r > 0$ $(1 \leq i \leq n)$, we have

$$|\Psi(\boldsymbol{\lambda}, \mathbf{t})| \prod_{1\leq i<j\leq n} |t_j^2 - t_i^2| \prod_{i=1}^n |\lambda_i|^{1/2} \prod_{1\leq i<j\leq n} |\lambda_i^2 - \lambda_j^2| \leq n! c_r e^{-r\langle\Re(\boldsymbol{\lambda}),\mathbf{t}_0\rangle},$$

where $\mathbf{t}_0 = (1, \ldots, 1)$ and c_r is a constant which depends only on r.

For all $\mathbf{t} \in \mathbb{R}^n$ we have

$$\begin{aligned} |\Psi(\boldsymbol{\lambda}, \mathbf{t})| \prod_{i=1}^n |\lambda_i|^{1/2} \prod_{1\le i<j\le n} |t_i^2 - t_j^2|\,|\lambda_i^2 - \lambda_j^2| &= \prod_{i=1}^n |\lambda_i|^{1/2} \Big| \det_{1\le i,j\le n} \Big(K_0(\lambda_i t_j)\Big)\Big| \\ &\le \sum_{\sigma\in\mathbb{S}_n} \prod_{i=1}^n |\lambda_i|^{1/2} \left|K_0(\lambda_i t_{\sigma(i)})\right|. \end{aligned}$$

It is well known that for $z \in \mathbb{C}\backslash]-\infty, 0]$ we have $K_0(z) = \sqrt{\frac{\pi}{2z}} W_{0,0}(2z)$, where $W_{0,0}$ denotes the Whittaker function. Using the expression (4), p. 317 of [36], and the asymptotic expression (1), p. 202 of [37], we get

$$|z|^{1/2}|K_0(z)| \le \text{const.}\, e^{-\Re(z)}, \qquad z \in \mathbb{C}\backslash]\infty, 0].$$

Thus, if $t_{\sigma(i)} \ge r > 0$ and $\lambda_i \in \mathbb{C}\backslash]-\infty, 0]$, then

$$|\lambda_i|^{1/2}|K_0(\lambda_i t_{\sigma(i)})| \le c_r e^{-\Re(\lambda_i) t_{\sigma(i)}}.$$

If in addition $\Re(\lambda_i) \ge 0$, we obtain

$$|\lambda_i|^{1/2}|K_0(\lambda_i t_{\sigma(i)})| \le c_r e^{-r\Re(\lambda_i)}.$$

The desired claim is now clear.

Recall that $\mathfrak{B}_R = \{\mathbf{t} \in \mathbb{R}^n \mid \|\mathbf{t}\| \le R\}$ with $R > 0$, and, for $0 < r < \infty$, set $\mathfrak{C}_r := \{\mathbf{t} \in \mathbb{R}^n \mid t_i \ge r \ (1 \le i \le n)\}$. Denote by $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ the space of functions $f \in \mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$ with support contained in $\mathfrak{C}_r \cap \mathfrak{B}_R$. Note that $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ} = \{0\}$ if $R \le r$. The union of the spaces $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ over all $0 < r < R < \infty$ coincides with $\mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$.

Claim 2. For all $0 < r < R < \infty$, the transformation $\widetilde{\mathscr{L}}$ maps $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ injectively into $\mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$.

Since the function $\boldsymbol{\lambda} \mapsto \Psi(\boldsymbol{\lambda}, \mathbf{t})$ is meromorphic on $\mathfrak{D}$ with simple poles at $\lambda_i + \lambda_j = 0$ for $1 \le i \ne j \le n$, it follows that $\boldsymbol{\lambda} \mapsto \widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda})$ extends to a meromorphic function on $\mathfrak{D}$ with simple poles at $\lambda_i + \lambda_j = 0$ for $i \ne j$. Further, the $\mathcal{W}_\circ$-invariance of the Bessel functions Ψ implies that $\boldsymbol{\lambda} \mapsto \widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda})$ is a $\mathcal{W}_\circ$-invariant map for all $f \in \mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$. Moreover, by means of Corollary 3.4, the Bessel-Laplace transform $\widetilde{\mathscr{L}}$ satisfies the property $(\mathbb{P}_1)$. One can also check that $\widetilde{\mathscr{L}}$ obeys the property $(\mathbb{P}_2)$. Indeed, for $f \in \mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ we have

$$\begin{aligned} &|\mathscr{L}^\circ(f)(\boldsymbol{\lambda})| \prod_{i=1}^n |\lambda_i|^{1/2} \prod_{1\le i<j\le n} |\lambda_i^2 - \lambda_j^2| \\ \le &\int_{\mathfrak{a}_-\cap\,\mathrm{supp}(f)} |f(\mathbf{t})|\ |\Psi(\boldsymbol{\lambda}, \mathbf{t})| \prod_{i=1}^n |\lambda_i|^{1/2} \prod_{1\le i<j\le n} |\lambda_i^2 - \lambda_j^2| \omega(\mathbf{t}) d\mathbf{t} \\ \le & c_{r,R}\, e^{-r\langle \Re(\boldsymbol{\lambda}), \mathbf{t}_0\rangle}. \end{aligned}$$

Above we used Claim 1. To reach the conclusion, it is enough to recall that $\Psi(\boldsymbol{\lambda}, \mathbf{t})$ satisfies a Bessel system of differential equations (cf. [11, (4.8)]).

The injectivity of $\widetilde{\mathscr{L}}$ follows from the inversion formula in Theorem 3.5.

Claim 3. If $\mathtt{av}(\widetilde{\mathscr{L}}(f))(\boldsymbol{\lambda}) = 0$ with $f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$, then $f \equiv 0$.

The statement of Corollary 3.4 can also be written as $\widetilde{\mathscr{F}}^{\mathrm{d}}(f^{\mathrm{d}})(\boldsymbol{\lambda}) = \frac{c(\mathrm{d})}{c(\Omega)}\mathtt{av}(\widetilde{\mathscr{L}}(f))(\boldsymbol{\lambda})$ where $f^{\mathrm{d}}_{|\mathfrak{a}_-} \equiv f_{|\mathfrak{a}_-}$. Thus, the claim is an easy consequence of the injectivity of the Bessel-Fourier transform $\widetilde{\mathscr{F}}^{\mathrm{d}}$.

The following property can be proved in a similar way as Lemma 9.1 in [2]. The function g in the statement bellow plays the same role as g_1 in the proof of [2, Lemma 9.1].

Claim 4. Let g be a meromorphic function on $\mathfrak{D}$ which satisfies the condition $(\mathbb{P}_2)$ for some $r > 0$. If $\mathtt{av}(g) \equiv 0$, then $g \equiv 0$.

Now we can state and prove the first main result of the paper, i.e. a Paley-Wiener theorem for $\widetilde{\mathscr{L}}$. Our approach is similar to the one used in [2] for the spherical Laplace transform $\mathscr{L}$.

Theorem A. *The Bessel-Laplace transform $\widetilde{\mathscr{L}}$ is a bijection of $\mathcal{C}_{r,R}^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$ onto $\mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$ for every $0 < r < R < \infty$, and of $\mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$ onto $\mathcal{PW}_\circ(\mathbb{C}^n)$.*

Proof. By virtue of Claim 2 we only need to prove the surjectivity of $\widetilde{\mathscr{L}}$ from $\mathcal{C}_{r,R}^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$ to $\mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$. From Theorem 3.1(i) we have

$$\begin{aligned}
\Phi(\boldsymbol{\lambda}, \mathbf{t}) &= \frac{\det\limits_{1\le i,j\le n}\big(I_0(\lambda_i t_j)\big)}{\prod\limits_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2)\prod\limits_{1\le i<j\le n}(t_j^2-t_i^2)} \\
&= \sum_{\sigma\in\mathbb{S}_n}(-1)^\sigma \frac{\prod\limits_{i=1}^n I_0\big(\lambda_{\sigma(i)}t_i\big)}{\prod\limits_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2)\prod\limits_{1\le i<j\le n}(t_j^2-t_i^2)} \\
&= \sum_{\sigma\in\mathbb{S}_n}\frac{1}{\prod\limits_{1\le i<j\le n}(t_j^2-t_i^2)}\frac{\prod\limits_{i=1}^n I_0\big(\lambda_{\sigma(i)}t_i\big)}{\prod\limits_{1\le i<j\le n}(\lambda_{\sigma(j)}^2-\lambda_{\sigma(i)}^2)} \\
&= \sum_{\sigma\in\mathbb{S}_n}\Xi(\sigma(\boldsymbol{\lambda}), \mathbf{t}),
\end{aligned}$$

where

$$\Xi(\boldsymbol{\lambda}, \mathbf{t}) := \frac{\prod\limits_{i=1}^n I_0(\lambda_i t_i)}{\prod\limits_{1\le i<j\le n}(t_j^2-t_i^2)\prod\limits_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2)},$$

with $t_i \ne \pm t_j$ and $\lambda_i \ne \pm\lambda_j$ for $i \ne j$.

For fixed r and R, we define the wave packet of $g \in \mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$ by

$$\mathscr{I}g(\mathbf{t}) = \int_{i\mathbb{R}^n} g(\boldsymbol{\lambda})\Phi(\boldsymbol{\lambda}, \mathbf{t})\omega(\boldsymbol{\lambda})\prod_{i=1}^n\frac{\lambda_i}{|\lambda_i|}d\boldsymbol{\lambda}$$

when $\mathbf{t} \in \mathfrak{a}_-$. The function $\mathscr{I} g$ is well defined and it belongs to $\mathcal{C}^\infty(c^\circ_{\max})^{\mathcal{W}_\circ}$. This follows from the growth behavior of $g \in \mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$, and the fact that

$$|\partial^{\alpha_1}_{t_1} \dots \partial^{\alpha_n}_{t_n} \Phi(\boldsymbol{\lambda}, \mathbf{t})| \le \text{const.}\, \|\boldsymbol{\lambda}\|^{\alpha_1+\dots+\alpha_n},$$

with $\boldsymbol{\lambda} \in i\mathbb{R}^n$ and $\mathbf{t} \in \mathbb{R}^n$. Here the constant "const." depends on $\alpha_1, \dots, \alpha_n$. Bellow we will prove that the support of $\mathscr{I} g$ is contained in $\mathfrak{C}_r \cap \mathfrak{B}_R$, i.e. $\mathscr{I} g \in \mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$.

By the $\mathcal{W}_\circ$-invariance of g and $\omega(\boldsymbol{\lambda})$, we have

$$\mathscr{I} g(\mathbf{t}) = n! \int_{i\mathbb{R}^n} g(\boldsymbol{\lambda}) \Xi(\boldsymbol{\lambda}, \mathbf{t}) \omega(\boldsymbol{\lambda}) \prod_{i=1}^n \frac{\lambda_i}{|\lambda_i|} d\boldsymbol{\lambda}, \qquad \mathbf{t} \in \mathfrak{a}_-.$$

On the other hand, using the expression of I_0 in [37, p. 77] and the asymptotic expression (2), p. 203 of [37], it follows that there exist two positive constants such that

$$\begin{aligned} &|I_0(z)| \le \text{const.}, && 0 \le |z| \le 1, \\ &|I_0(z)| \le \text{const.}\, |z|^{-1/2} e^{\Re(z)}, && 1 \le |z|. \end{aligned}$$

Thus, for fixed $\mathbf{t} \in \mathfrak{a}_-$,

$$|\Xi(\boldsymbol{\lambda}, \mathbf{t})| \omega(\boldsymbol{\lambda})^{1/2} \le \text{const.} \frac{1}{\displaystyle\prod_{i=1}^n t_i^{1/2} \prod_{1 \le i < j \le n} (t_j^2 - t_i^2)} \tag{4.1}$$

if $|\lambda_i| \le t_i^{-1}$ for all i, and

$$|\Xi(\boldsymbol{\lambda}, \mathbf{t})| \omega(\boldsymbol{\lambda})^{1/2} \le \text{const.} \frac{e^{\langle \Re(\boldsymbol{\lambda}), \mathbf{t} \rangle}}{\displaystyle\prod_{i=1}^n t_i^{1/2} \prod_{1 \le i < j \le n} (t_j^2 - t_i^2)} \tag{4.2}$$

if $|\lambda_i| \ge t_i^{-1}$ for all i.

Now let $\mathbf{t} \in \mathfrak{a}_- \backslash \mathfrak{C}_r$. By [2, p. 721], there exists an element $\boldsymbol{\lambda}^\circ \in \mathbb{R}^n_+$ such that $\zeta := \langle \boldsymbol{\lambda}^\circ, \mathbf{t} - r\mathbf{t}_\circ \rangle < 0$, where $\mathbf{t}_\circ = (1, \dots, 1)$. Hence, for arbitrary $\alpha \gg 0$, we have

$$\begin{aligned} |\Xi(\boldsymbol{\lambda} + \alpha\boldsymbol{\lambda}^\circ, \mathbf{t})|\, \omega(\boldsymbol{\lambda} + \alpha\boldsymbol{\lambda}^\circ)^{\frac{1}{2}} &= \frac{\displaystyle\prod_{i=1}^n |\lambda_i + \alpha\lambda_i^\circ|^{1/2} \prod_{i=1}^n \left| I_0\big((\lambda_i + \alpha\lambda_i^\circ) t_i\big)\right|}{\displaystyle\prod_{1 \le i < j \le n} (t_j^2 - t_i^2)} \\ &\sim \frac{e^{\langle \Re(\boldsymbol{\lambda} + \alpha\boldsymbol{\lambda}^\circ), \mathbf{t} \rangle}}{\displaystyle\prod_{i=1}^n t_i^{1/2} \prod_{1 \le i < j \le n} (t_j^2 - t_i^2)} \qquad \text{as } \alpha \to \infty. \end{aligned}$$

Above we used the fact that

$$I_0(z) \sim z^{-1/2} e^z \qquad \text{as } z \to \infty.$$

In particular, if $\boldsymbol{\lambda} \in i\mathbb{R}^n$ and $\mathbf{t} \in \mathfrak{a}_- \backslash \mathfrak{C}_r$, then there exists a constant not depending on $\boldsymbol{\lambda}$ such that $|\Xi(\boldsymbol{\lambda} + \alpha\boldsymbol{\lambda}^\circ, \mathbf{t})|\, \omega(\boldsymbol{\lambda} + \alpha\boldsymbol{\lambda}^\circ)^{1/2}$ is bounded by $c e^{\alpha \langle \boldsymbol{\lambda}^\circ, \mathbf{t} \rangle}$ as α goes to infinity. That is

$$|\Xi(\boldsymbol{\lambda} + \alpha\boldsymbol{\lambda}^\circ, \mathbf{t})|\, \omega(\boldsymbol{\lambda} + \alpha\boldsymbol{\lambda}^\circ)^{1/2} \le c e^{\alpha\zeta} e^{r\alpha \langle \boldsymbol{\lambda}^\circ, \mathbf{t}_\circ \rangle} \qquad \text{as } \alpha \to \infty. \tag{4.3}$$

By virtue of (4.1), (4.2), (4.3), and the growth behavior of $g \in \mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$, Cauchy's theorem and a contour shift imply that

$$\begin{aligned}
\mathscr{I}g(\mathbf{t}) &= n! \int_{i\mathbb{R}^n} g(\boldsymbol{\lambda})\Xi(\boldsymbol{\lambda},\mathbf{t})\omega(\boldsymbol{\lambda})\prod_{i=1}^n \frac{\lambda_i}{|\lambda_i|}d\boldsymbol{\lambda} \\
&= n! \int_{i\mathbb{R}^n} g(\boldsymbol{\lambda}+\alpha\boldsymbol{\lambda}^\circ)\Xi(\boldsymbol{\lambda}+\alpha\boldsymbol{\lambda}^\circ,\mathbf{t})\omega(\boldsymbol{\lambda}+\alpha\boldsymbol{\lambda}^\circ)\prod_{i=1}^n \frac{\lambda_i+\alpha\lambda_i^\circ}{|\lambda_i+\alpha\lambda_i^\circ|}d\boldsymbol{\lambda} \\
&\quad \longrightarrow 0 \qquad \text{as } \alpha\to\infty.
\end{aligned}$$

Thus $\mathscr{I}g$ vanishes on $\mathfrak{a}_- \setminus \mathfrak{C}_r$, and, by the continuity and the $\mathcal{W}_\circ$-invariance of $\mathscr{I}g$, this is equivalent to $\mathscr{I}g \equiv 0$ on $c_{\max}^\circ \setminus \mathfrak{C}_r$. Furthermore, the wave packet vanishes also on $c_{\max}^\circ \setminus \mathfrak{B}_R$. One can see this as follows: Recall that $\mathcal{W}_\circ \setminus \mathcal{W} \cong \{\pm 1\}^n$. Using the $\mathcal{W}$-invariance of Φ and $\omega(\boldsymbol{\lambda})$, we have (for $\mathbf{t} \in \mathfrak{a}_-$)

$$\begin{aligned}
\mathscr{I}g(\mathbf{t}) &= \int_{i\mathbb{R}^n} g(\boldsymbol{\lambda})\Phi(\boldsymbol{\lambda},\mathbf{t})\omega(\boldsymbol{\lambda})\prod_{i=1}^n \frac{\lambda_i}{|\lambda_i|}d\boldsymbol{\lambda} \\
&= \frac{1}{2^n}\int_{i\mathbb{R}^n} \sum_{\tau\in\{\pm1\}^n}\prod_{i=1}^n \{(-\tau_i\lambda_i)^{-1/2}(\tau_i\lambda_i)^{1/2}\} g(\tau\boldsymbol{\lambda})\Phi(\boldsymbol{\lambda},\mathbf{t})\omega(\boldsymbol{\lambda})d\boldsymbol{\lambda} \\
&= \frac{1}{2^n}\frac{c(\Omega)}{c(\mathrm{d})}\int_{i\mathbb{R}^n} \mathtt{av}(g)(\boldsymbol{\lambda})\Phi(\boldsymbol{\lambda},\mathbf{t})\omega(\boldsymbol{\lambda})d\boldsymbol{\lambda}.
\end{aligned}$$

Comparing this integral formula with (3.3), we get (up to a positive constant which does not depend on $\boldsymbol{\lambda}$)

$$\widetilde{\mathscr{F}}^{\mathrm{d}}(\mathscr{I}g)(\boldsymbol{\lambda}) = \text{const.}\, \mathtt{av}(g)(\boldsymbol{\lambda}). \tag{4.4}$$

Since $g \in \mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$, the property $(\mathbb{P}_1)$ implies that $\widetilde{\mathscr{F}}^{\mathrm{d}}(\mathscr{I}g)$ belongs to the Paley-Wiener space $\mathcal{H}_{\mathcal{W}}^R(\mathbb{C}^n)$. Hence, by Theorem 4.1, we conclude that $\mathrm{Supp}(\mathscr{I}g) \subset \mathfrak{B}_R$, i.e. $\mathscr{I}g(\mathbf{t}) = 0$ for all $\mathbf{t} \in c_{\max}^\circ \setminus \mathfrak{B}_R$. Thus we draw the conclusion that $\mathscr{I}g \in \mathcal{C}_{r,R}^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$. Moreover, by Corollary 3.4, equation (4.4) yields

$$\frac{c(\mathrm{d})}{c(\Omega)}\mathtt{av}(\widetilde{\mathscr{L}}(\mathscr{I}g))(\boldsymbol{\lambda}) = \widetilde{\mathscr{F}}^{\mathrm{d}}(\mathscr{I}g)(\boldsymbol{\lambda}) = \text{const.}\, \mathtt{av}(g)(\boldsymbol{\lambda}),$$

for all $g \in \mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$. Now, Claim 4 implies that (up to a constant) $\widetilde{\mathscr{L}}(\mathscr{I}(g)) = g$ for all $g \in \mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$. This finishes the proof of Theorem A. □

5. The inverse flat Abel transform for $SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$

Replacing the Cartan involution by the involution σ in the proof of [21, Theorem I.5.17], one can prove that for $f \in \mathcal{C}_c(C_{\max}^\circ)$

$$\int_{C_{\max}^\circ} f(Y)dY = \text{const.} \int_H \int_{\mathfrak{a}_-} f(\mathrm{Ad}(h)X) \prod_{\alpha\in\Sigma^+} |\langle \alpha, X\rangle|^{m_\alpha} dh dX,$$

where "const." is some positive constant depending only on the normalization of the measures. Thus, for $\boldsymbol{\lambda} \in \mathfrak{a}_{\mathbb{C}}^*$ such that $\Re(\lambda_i) > 0$ $(1 \le i \le n)$, the Bessel-Laplace transform of all $f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ} \cong \mathcal{C}_c^\infty(C_{\max}^\circ)^{\mathrm{Ad}(H)}$ can be written as

$$\begin{aligned}
\widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda}) &= \int_{\mathfrak{a}_-} f(X)\Psi(\boldsymbol{\lambda}, X)\omega(X)dX \\
&= \int_{\mathfrak{a}_-} f(X)\Big(\int_H e^{-\boldsymbol{\lambda}(\mathrm{Ad}(h)X)}dh\Big)\omega(X)dX \\
&= \text{const.} \int_{C_{\max}^\circ} f(X)e^{-\boldsymbol{\lambda}(X)}dX.
\end{aligned}$$

Above we used the following integral representation

$$\Psi(\boldsymbol{\lambda}, X) = \int_H e^{-\boldsymbol{\lambda}(\mathrm{Ad}(h)X)}dh.$$

We refer to [11, Theorem 4.12] for more details on the integral representation of $\Psi(\boldsymbol{\lambda}, X)$. Denote by $\mathfrak{a}^\perp$ the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{q}$. Thus, for $\boldsymbol{\lambda} \in \mathfrak{a}^*$ such that $\lambda_i > 0$ $(1 \le i \le n)$, we have

$$\begin{aligned}
\widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda}) &= \text{const.} \int_{C_{\max}^\circ \cap \mathfrak{a}} e^{-\boldsymbol{\lambda}(X)}\Big(\int_{C_{\max}^\circ \cap \mathfrak{a}^\perp} f(X+Y)dY\Big)dX \\
&= \text{const.} \int_{c_{\max}^\circ} e^{-\boldsymbol{\lambda}(X)}\mathscr{A}(f)(X)dX, \qquad (5.1)
\end{aligned}$$

where

$$\mathscr{A}(f)(X) := \int_{C_{\max}^\circ \cap \mathfrak{a}^\perp} f(X+Y)dY$$

denotes the "flat Abel" transform of $f \in \mathcal{C}_c^\infty(C_{\max}^\circ)^{\mathrm{Ad}(H)} \cong \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$ at $X \in c_{\max}^\circ$. The expression (5.1) is an analogue to the one proved by Helgason in [19] for the Bessel-Fourier transform on $\mathfrak{p}$. It follows that

$$\widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda}) = \text{const.} \int_{c_{\max}^\circ} e^{-\boldsymbol{\lambda}(X)}\mathscr{A}(f)(X)dX = \text{const.}\, \mathscr{F}\big(\mathscr{A}(f)\big)(\boldsymbol{\lambda}), \qquad (5.2)$$

where $\mathscr{F}$ stands for the Euclidean-Laplace transform associated with $c_{\max}^\circ$. In particular, if we set $\mathbb{V}(x_1, \dots, x_n) := \prod_{1 \le i < j \le n}(x_j^2 - x_i^2)$, then there are two ways of writing $\mathbb{V}(\lambda_1, \dots, \lambda_n)\widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda})$. First, by (5.2), we have

$$\begin{aligned}
\mathbb{V}(\lambda_1, \dots, \lambda_n)\widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda}) &= \text{const.}\, \mathbb{V}(\lambda_1, \dots, \lambda_n)\mathscr{F}\big(\mathscr{A}(f)\big)(\boldsymbol{\lambda}) \\
&= \text{const.}\, \mathscr{F}\Big[\mathbb{V}(\partial_1, \dots, \partial_n)\mathscr{A}(f)\Big](\boldsymbol{\lambda}). \qquad (5.3)
\end{aligned}$$

Second, for every $f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$, we have

$$\begin{aligned}
\mathbb{V}(\lambda_1,\ldots,\lambda_n)\widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda}) &= \text{const.} \int_{\mathfrak{a}_-} f(\mathbf{t}) \det_{1\le i,j\le n}\big(K_0(\lambda_i t_j)\big) \frac{\omega(\mathbf{t})}{\prod\limits_{1\le i<j\le n}(t_j^2-t_i^2)} d\mathbf{t} \\
&= \text{const.} \int_{\mathfrak{a}_-} f(\mathbf{t}) \det_{1\le i,j\le n}\big(K_0(\lambda_i t_j)\big) \prod_{i=1}^n t_i \prod_{1\le i<j\le n}(t_j^2-t_i^2) d\mathbf{t} \\
&= \text{const.} \sum_{\sigma\in\mathbb{S}_n} \int_{\mathfrak{a}_-} f(\mathbf{t}) \prod_{i=1}^n t_{\sigma(i)} K_0\big(\lambda_i t_{\sigma(i)}\big) \prod_{1\le i<j\le n}(t_{\sigma(j)}^2 - t_{\sigma(i)}^2) d\mathbf{t} \\
&= \text{const.} \int_{c_{\max}^\circ} f(\mathbf{t}) \mathbb{V}(t_1,\ldots,t_n) \prod_{i=1}^n t_i \prod_{i=1}^n K_0\big(\lambda_i t_i\big) d\mathbf{t}. \qquad (5.4)
\end{aligned}$$

Moreover, for $z,\nu \in \mathbb{C}$ such that $\Re(z) > 0$ and $\Re(\nu) > -1/2$, we have

$$K_\nu(z) = \frac{\sqrt{\pi} z^\nu}{2^\nu \Gamma(\nu+1/2)} \int_0^\infty e^{-z\,\mathrm{ch}(u)} \,\mathrm{sh}^{2\nu}(u) du.$$

A change of variable implies that

$$K_\nu(zt) = \frac{\sqrt{\pi}}{2^\nu \Gamma(\nu+1/2)} (zt^{-1})^\nu \int_t^\infty e^{-zs}(s^2-t^2)^{\nu-1/2} ds$$

with $t > 0$. In particular, for $\Re(z) > 0$ and $t > 0$,

$$K_0(zt) = \int_t^\infty \frac{e^{-zs}}{\sqrt{s^2-t^2}} ds.$$

Substituting the above integral representation of $K_0(\lambda_i t_i)$ in (5.4), we get

$$\begin{aligned}
&\mathbb{V}(\lambda_1,\ldots,\lambda_n)\widetilde{\mathscr{L}}(f)(\boldsymbol{\lambda}) \\
= \;& \text{const.} \int_{c_{\max}^\circ} f(t_1,\ldots,t_n)\mathbb{V}(t_1,\ldots,t_n) \prod_{i=1}^n t_i \Big\{ \int_{t_1}^\infty \cdots \int_{t_n}^\infty \prod_{i=1}^n \frac{e^{-\lambda_i s_i}}{\sqrt{s_i^2-t_i^2}} ds_i \Big\} d\mathbf{t} \\
= \;& \text{const.} \int_{c_{\max}^\circ} \prod_{i=1}^n e^{-\lambda_i s_i} \Big[\int_0^{s_1} \cdots \int_0^{s_n} f(t_1,\ldots,t_n)\mathbb{V}(t_1,\ldots,t_n) \prod_{i=1}^n \frac{t_i}{\sqrt{s_i^2-t_i^2}} dt_i \Big] d\mathbf{s} \\
= \;& \text{const.}\, \mathscr{F}\Big(\mathbb{A}_1^{\otimes n}(f\mathbb{V})\Big)(\boldsymbol{\lambda}), \qquad (5.5)
\end{aligned}$$

where $\mathbb{A}_1^{\otimes n}$ denotes the n-fold tensor product of the one dimensional integral transformation

$$\mathbb{A}_1(F)(s) := \int_0^s F(t) \frac{t}{\sqrt{s^2-t^2}} dt, \qquad F \in \mathcal{C}_c^\infty(\mathbb{R}^+),\; s > 0.$$

The later transform satisfies

$$F(t) = \text{const.}\, \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathbb{A}_1(F)(s) \frac{s}{\sqrt{t^2-s^2}} ds. \qquad (5.6)$$

Comparing (5.3) with (5.5), and using the injectivity of the Euclidean-Laplace transform $\mathscr{F}$, we obtain

$$\mathbb{V}(\partial_1,\ldots,\partial_n)\mathscr{A}(f)(\mathbf{t}) = \text{const.}\, \mathbb{A}_1^{\otimes n}(f\mathbb{V})(\mathbf{t}).$$

In view of (5.6), we obtain the second main result of the paper.

Theorem B. *Assume that $f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$. For every $\mathbf{t} \in \mathfrak{a}_-$, the inverse Abel transform is expressed as*

$$\mathbb{V}(t_1, \dots, t_n) f(\mathbf{t}) = \text{const.} \prod_{i=1}^n \frac{1}{t_i} \frac{\mathrm{d}}{\mathrm{d}t_i} \int_0^{t_1} \cdots \int_0^{t_n} \mathbb{V}(\partial_1, \dots, \partial_n) \mathscr{A}(f)(\mathbf{s}) \prod_{i=1}^n \frac{s_i}{\sqrt{t_i^2 - s_i^2}} d\mathbf{s},$$

where $\mathbb{V}(x_1, \dots, x_n) = \prod_{1 \le i < j \le n} (x_j^2 - x_i^2)$.

***Remark* 5.1.** (*Another way of computing the Bessel function $\Psi(\boldsymbol{\lambda}, \mathbf{t})$ via the rank one case*) Denote by $\mathcal{M}_{(1,0)}^{(1)}$ the rank one symmetric space $SO_0(1,2)/SO_0(1,1)$. The associated restricted root system is given by $\{\pm\alpha\}$, where $\alpha(t) = -t$ defines the positive root. Here $\mathfrak{a} \cong \mathbb{R}$, and $m_\alpha = 1$. By [11, Example 4.13], the Bessel functions associated with $\mathcal{M}_{(1,0)}^{(1)}$ are given by

$$\Psi_{(1,0)}^{(1)}(\lambda, t) = K_0(\lambda t), \qquad \Re(\lambda) > 0,\ t > 0.$$

Let $\mathcal{M}_{(1,0)}^{(n)}$ be the product of n-copies of $\mathcal{M}_{(1,0)}^{(1)}$, and define on $\mathcal{M}_{(1,0)}^{(n)}$ the pseudo-Bessel function

$$\Psi_{(1,0)}^{(n)}(\boldsymbol{\lambda}, \mathbf{t}) := \sum_{\sigma \in \mathbb{S}_n} \prod_{i=1}^n K_0(\lambda_{\sigma(i)} t_i).$$

On the other hand, recall that the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ associated with $\mathcal{M} = SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_*^+$ consists of long roots with multiplicities 1 and short roots with multiplicities 2. By [30] we can prove that we may obtain the Bessel function $\Psi(\boldsymbol{\lambda}, \mathbf{t})$ associated with $\mathcal{M}$ via $\Psi_{(1,0)}^{(n)}(\boldsymbol{\lambda}, \mathbf{t})$ as followsing:

$$\Psi(\boldsymbol{\lambda}, , \mathbf{t}) = \frac{\text{const.}}{\displaystyle\prod_{1 \le i < j \le n} (\lambda_j^2 - \lambda_i^2)^2} \mathscr{D}_{(0,2)} \Psi_{(1,0)}^{(n)}(\boldsymbol{\lambda}, \mathbf{t}), \tag{5.7}$$

where $\mathscr{D}_{(0,2)}$ denotes the shift operator

$$\prod_{1 \le i < j \le n} (t_j^2 - t_i^2)^{-1} \prod_{1 \le i < j \le n} \big(\mathscr{L}(t_j, \partial_{t_j}) - \mathscr{L}(t_i, \partial_{t_i})\big),$$

with

$$\mathscr{L} := \frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt}.$$

Using the fact that $K_\nu(z)$ is a solution to

$$u'' + \frac{1}{z}u' - \Big(1 + \frac{\nu^2}{z^2}\Big)u = 0,$$

we deduce that

$$\begin{aligned}
\Psi(\boldsymbol{\lambda},\mathbf{t}) &= \frac{\text{const.}}{\prod\limits_{1\le i<j\le n}(t_j^2-t_i^2)(\lambda_j^2-\lambda_i^2)^2}\sum_{\sigma\in\mathbb{S}_n}\prod_{i=1}^n K_0(\lambda_{\sigma(i)}t_i)\prod_{1\le i<j\le n}(\lambda_{\sigma(j)}^2-\lambda_{\sigma(i)}^2)\\
&= \frac{\text{const.}}{\prod\limits_{1\le i<j\le n}(t_j^2-t_i^2)(\lambda_j^2-\lambda_i^2)}\sum_{\sigma\in\mathbb{S}_n}(-1)^\sigma\prod_{i=1}^n K_0(\lambda_{\sigma(i)}t_i)\\
&= \text{const.}\,\frac{\det\limits_{1\le i,j\le n}\Big(K_0(\lambda_i t_j)\Big)}{\prod\limits_{1\le i<j\le n}(t_j^2-t_i^2)\prod\limits_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2)},
\end{aligned}$$

which coincides with Theorem 3.1(ii). Notice that one may use formula (5.7) to give another proof for Theorem A. In a forthcoming paper we shall develop this approach further to prove a Paley-Wiener theorem for a larger class of noncompact causal symmetric spaces.

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